

From the Boltzmann H -theorem to Perelman's W -entropy formula for the Ricci flow*

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Abstract. In 1870s, L. Boltzmann proved the famous H -theorem for the Boltzmann equation in the kinetic theory of gas and gave the statistical interpretation of the thermodynamic entropy. In 2002, G. Perelman introduced the notion of W -entropy and proved the W -entropy formula for the Ricci flow. This plays a crucial role in the proof of the no local collapsing theorem and in the final resolution of the Poincaré conjecture and Thurston's geometrization conjecture. In our previous paper [21], the author gave a probabilistic interpretation of the W -entropy using the Boltzmann-Shannon-Nash entropy. In this paper, we make some further efforts for a better understanding of the mysterious W -entropy by comparing the H -theorem for the Boltzmann equation and the Perelman W -entropy formula for the Ricci flow. We also suggest a way to construct the “density of states” measure for which the Boltzmann H -entropy is exactly the W -entropy for the Ricci flow.

1 Introduction

In a seminal paper [31], Perelman introduced the W -entropy for the Ricci flow and the associated conjugate heat equation and proved its monotonicity on compact manifolds. More precisely, let M be an n -dimensional compact manifold with a family of Riemannian metrics $\{g(t) : t \in [0, T]\}$, $f \in C^\infty(M \times [0, T], \mathbb{R})$, where $T > 0$ is fixed. Suppose that $(g(t), f(t), \tau(t), t \in [0, T])$ is a solution to the evolution equations

$$\partial_t g = -2Ric, \quad (1)$$

$$\partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \quad (2)$$

$$\partial_t \tau = -1, \quad (3)$$

Following [31], the W -entropy functional associated to (1) and (2) is defined by

$$W(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv, \quad (4)$$

where v denotes the volume measure on $(M, g(\tau))$. By [31], the following entropy formula holds

$$\frac{d}{dt} W(g, f, \tau) = 2 \int_M \tau \left| Ric + \nabla^2 f - \frac{g}{2\tau} \right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv. \quad (5)$$

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As a consequence of the Perelman W -entropy formula (5), the W -entropy is monotone increasing in τ and the monotonicity is strict except that $(M, g(\tau), f(\tau))$ is a shrinking Ricci soliton, i.e.,

$$Ric + \nabla^2 f = \frac{g}{2\tau}. \quad (6)$$

This result plays a crucial role in the proof of the no local collapsing theorem, which is equivalent to the long time standing Hamilton's Little Loop Conjecture [18]. As pointed out by Perelman [31], "this removes the major stumbling block in Hamilton's approach to geometrization" (for the resolution of the Poincaré conjecture and Thurston's geometrization conjecture). See also [4, 12, 19, 25].

Since Perelman's paper [31] was posted on Arxiv in 2002, many people have established the W -entropy formula for various geometric evolution equations [5, 21, 22, 28, 29, 16, 24, 20]. However, it seems that there is no reference in the literature which explains clearly how the mysterious W -entropy was introduced for the Ricci flow and what are the hidden insights for Perelman to introduce the W -entropy for the Ricci flow. In our previous paper [21], the author gave a probabilistic interpretation of the W -entropy using the Boltzmann-Shannon-Nash entropy. The purpose of this paper is try to make some further efforts for a better understanding of the mysterious W -entropy by comparing the H -theorem for the Boltzmann equation and the Perelman W -entropy formula for the Ricci flow. We also suggest a way to construct the "density of states" measure for which the Boltzmann H -entropy is exactly the W -entropy for the Ricci flow.

2 History and Boltzmann entropy formula

2.1 Thermodynamic entropy

The notion of entropy was introduced by R. Clausius in 1865 in his study of the Carnot cycle in the thermodynamics [13, 14]. He proved that, in any reversible thermodynamic process, $\frac{\delta Q}{T}$ is an exact form, where δQ denotes the total change of the heat along the reversible thermodynamic process, and T is the temperature. This leads Clausius to define the thermodynamic entropy S as a state function of the thermodynamic system satisfying

$$dS = \frac{\delta Q}{T}.$$

Thus, if the reversible thermodynamic process changes its states from A to B , then

$$S(B) - S(A) = \int_A^B \frac{\delta Q}{T}.$$

Moreover, the value of $S(B) - S(A)$ is independent of the choice of any reversible process from A to B . If the thermodynamic process is irreversible, the second law of thermodynamics says that: The thermodynamic entropy S of an isolated thermodynamic system always increases, i.e.,

$$\Delta S \geq 0.$$

In this sense, the thermodynamic entropy S is an important quantity (as a function of the state) to characterize the irreversibility of the thermodynamic processes.

2.2 Boltzmann equation and H -theorem

In 1872, Boltzmann published an important paper [1], which contained two celebrated results nowadays known as the Boltzmann equation and the Boltzmann H -theorem. More precisely,

let $f(x, v, t)$ be the probabilistic distribution of the ideal gas at time t in the phase space \mathbb{R}^6 of position x and velocity v , then f satisfies the Boltzmann equation (see e.g. [1, 3, 17, 9, 8, 10])

$$\partial_t f + v \cdot \nabla_x f = Q(f, f)$$

where

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} [f(x, v') f(x, v'_*) - f(x, v) f(x, v_*)] B(v - v_*, \theta) dv_* dS(u),$$

where dS denotes the surface measure on S^2 , v', v'_* are defined in terms of v, v_*, u by

$$v' = v - [(v - v_*) \cdot u]u, \quad v'_* = v_* + [(v - v_*) \cdot u]u,$$

and $B : \mathbb{R}^3 \times S^2 \rightarrow (0, \infty)$ is the collision kernel and is assumed to be rotationally invariant, i.e., $B(z, u) = B(|z|, |z \cdot u|)$, $\forall z \in \mathbb{R}^3$ and $u \in S^2$. Let

$$H(t) = \int_{\mathbb{R}^6} f(x, v, t) \log f(x, v, t) dx dv \quad (7)$$

be the Boltzmann H -functional of the probability distribution $f(x, v, t) dx dv$. The Boltzmann H -Theorem states that, if $f(x, v, t)$ is the solution of the Boltzmann equation, which is “sufficiently well-behaved” (in the sense of Villani [34], for its precise meaning, see [9]), then the following formula holds (see e.g. [1, 3, 17, 9, 8, 10])

$$\frac{dH}{dt} = \frac{1}{4} \int_{S^2} \int_{\mathbb{R}^3} (f' f'_* - f f_*) [\log(f f_*) - \log(f' f'_*)] B(v - v_*, \theta) dv_* dS(u), \quad (8)$$

where $f = f(\cdot, v, \cdot)$, $f' = f(\cdot, v', \cdot)$, $f_* = f(\cdot, v_*, \cdot)$, $f'_* = f(\cdot, v'_*, \cdot)$. By the Boltzmann formula for the H -functional (8) and using the elementary inequality

$$(x - y)(\log y - \log x) \leq 0, \quad \forall x, y \in \mathbb{R}^+,$$

one can conclude that H is always decreasing in time, i.e.,

$$\frac{dH}{dt} \leq 0, \quad \forall t > 0, \quad (9)$$

and the equality holds in (9) if and only if

$$f' f'_* = f f_*, \quad \forall v, v_* \in \mathbb{R}^3, u \in S^2, \quad (10)$$

which implies that f is a local Maxwell distribution, i.e.,

$$f(x, v, t) = n(x, t) \left(\frac{m}{2\pi kT(x, t)} \right)^{3/2} \exp \left(-\frac{|v - \bar{v}(x, t)|^2}{2kmT(x, t)} \right), \quad (11)$$

where the parameters m is the mass of particle, $n(x, t) \in \mathbb{R}^3$ is the particle density at (x, t) , $\bar{v}(x, t) \in \mathbb{R}^3$ and $T(x, t) > 0$ are the mean velocity and the local temperature at (x, t) .

2.3 Boltzmann entropy formula

In 1877, to better explain his H -theorem, Boltzmann [2] introduced the statistical interpretation of the thermodynamic entropy S by the formula

$$S = k \log W, \quad (12)$$

where k is a constant, which is nowadays called the Boltzmann constant (for simplicity, we choose $k = 1$ throughout this paper), and W is the German word “Wahrscheinlichkeit” of the English word “probability”, which is identified with the number of possible microstates corresponding to the macroscopic state of a given thermodynamic system, i.e., the number of (unobservable) “way” in the (observable) thermodynamic state of a system can be realized by assigning different positions and momenta to various molecules of the ideal gas. The Boltzmann entropy formula (12) indicates the logarithmic connection between Clausius’ thermodynamic entropy S and the number W of the most probable microstates consistent with the given macrostate.

We now explain the Boltzmann entropy formula (12) by studying the following question: Take N identical particles, which will be distributed over k boxes, and let p_1, \dots, p_k be some rational numbers in $[0, 1]$ such that $\sum_{i=1}^k p_i = 1$. We can regard p_j as the probability for each particle to be distributed into the j -th box. Let W be the number of configurations such that N_j particles are distributed in the j -th box, $j = 1, \dots, k$, and P the corresponding probability. Then

$$W = \frac{N!}{N_1! \dots N_k!},$$

and

$$P = \frac{N!}{N_1! \dots N_k!} p_1^{N_1} \dots p_k^{N_k}.$$

Assuming that $N_j \gg 1$, $j = 1, \dots, k$. By the Stirling formula $\log N! \equiv (N + 1/2) \log N + N + \frac{1}{2} \log(2\pi) + O(1/N)$, we have

$$\log P \simeq \left(N + \frac{1}{2}\right) \log N - \sum_{i=1}^k \left(N_j + \frac{1}{2}\right) \log N_j - \sum_{j=1}^k N_j \log p_j - \frac{k-1}{2} \log(2\pi).$$

Using the method of the Lagrangian multiplier, we can prove that the maximum value of P subject to the constraint $\sum_{j=1}^k N_j = N$ is achieved at $N_j = p_j N$ and we have

$$\log P_{\max} \simeq -\frac{1}{2} \log N - \sum_{i=1}^k \left(2p_j N + \frac{1}{2}\right) \log p_j - \frac{k-1}{2} \log(2\pi).$$

Therefore

$$\lim_{N \rightarrow \infty} \frac{\log P_{\max}}{N} = -2 \sum_{i=1}^k p_i \log p_i.$$

Let W_{\max} be the corresponding number of the configurations such that P achieves its maximum value P_{\max} . Then (cf. also [34])

$$S := \lim_{N \rightarrow \infty} \frac{\log W_{\max}}{N} = - \sum_{i=1}^k p_i \log p_i. \quad (13)$$

By comparing the Boltzmann entropy formula (13) with the H -functional (7) introduced by Boltzmann in the study of the kinetic theory of gas, one can see that the H -functional is nothing else but the minus entropy of the probability distribution $f(\cdot, \cdot, t)$ in the phase space at time t .

2.4 Shannon and Nash entropy

In 1948, C. Shannon [32] introduced the notion of entropy into the theory of information. More precisely,

$$H = - \sum_{i=1}^k p_i \log p_i.$$

According to Shannon [32], “the form of H will be recognized as that of entropy as defined in certain formulations of statistical mechanics where p_i is the probability of a system being in cell i of its phase space... H is then, for example, the H -in Boltzmanns famous H -theorem.”

On the other hand, J. Nash [26] used the H -entropy to study the continuity of the parabolic and elliptic PDEs. In the literature, the Boltzmann H -entropy is sometime called the Shannon entropy or the Nash entropy. In this paper, we call H the Boltzmann-Shannon-Nash entropy, or for simplicity the Boltzmann entropy.

2.5 The maximum entropy principle and the central limit theorem

According to Boltzmann’s entropy formula, the entropy S of a given macrostate achieves its maximum value at the most probable microstates subject to the constraint $\sum_{j=1}^k N_j = N$.

The following theorem, which can be proved by the method of Lagrangian multiplier, indicates a deep connection between the central limit theorem and the maximum entropy principle, in view of which, one can interpret the central limit theorem as a consequence of the maximum entropy principle.

Theorem 2.1 *Let $\mathcal{D} = \{\mu = f dx \in \mathbb{P}(\mathbb{R}) : \int_M x d\mu(x) = 0, \int_M x^2 d\mu(x) = 1\}$. Then*

$$\gamma = \arg \min \{H(\mu) : \mu \in \mathcal{D}\}.$$

2.6 Canonical ensemble and Boltzmann’s entropy formula

Heuristically, the canonical ensemble is a measurable configuration space (Ω, \mathcal{F}) on which “there exist” *a priori measure* $D\omega$ and a Hamiltonian function $E : \Omega \rightarrow [0, +\infty]$, such that the distribution of the particles on (Ω, \mathcal{F}) is given by a probability measure \mathbb{P} on (Ω, \mathcal{F}) which has the following formal expression

$$d\mathbb{P}(\omega) = \frac{1}{Z_\beta} e^{-\beta E(\omega)} D\omega,$$

where

$$Z_\beta = \int_{\Omega} e^{-\beta E(\omega)} D\omega,$$

is called the partition function. In the literature, the probability measure \mathbb{P} is called the Gibbs measure. We use $(\Omega, \mathcal{F}, E, \mathbb{P})$ to denote the canonical ensemble (Ω, \mathcal{F}) with the Hamiltonian function E and with the Gibbs measure \mathbb{P} . In the kinetic theory of ideal gas, Ω is the phase space of the positions and the velocities of the particles, and \mathbb{P} is also called the Maxwell-Boltzmann distribution.

To introduce the Maxwell-Boltzmann distribution, let us consider the following problem: Let us assume that N particles of ideal gas are distributed into n rooms R_1, \dots, R_n , each room R_i has g_i boxes, $i = 1, \dots, n$. Suppose that if a particle is in the room R_i , it has energy E_i , $i = 1, \dots, n$. Then the total number of ways that these N particles are distributed in

to boxes of these n rooms, for which N_i particles are in the room R_i and with energy E_i , is given by

$$W = \frac{N!}{\prod_{i=1}^n N_i!} \prod_{i=1}^n g_i^{N_i}.$$

The most probable distribution is the one for which W achieves its maximum value under two constraint conditions:

$$N = \sum_{i=1}^n N_i, \quad E = \sum_{i=1}^n N_i E_i.$$

By the method of Lagrange multiplier, and using the Stirling asymptotic formula, one can prove that, in the case $N \gg 1$ and $N_i \gg 1$, the most probable distribution satisfying the above two constraint conditions is given by

$$N_i = A g_i e^{-\beta E_i}, \quad i = 1, \dots, n.$$

where A and β are two constants. By the theory of thermodynamic, it is known that β is related to the absolute temperature T via

$$\beta = \frac{1}{kT},$$

where k is a constant, called the Boltzmann constant. To determine A , it is useful to introduce the partition function

$$Z_\beta = \sum_{i=1}^n g_i e^{-\beta E_i}.$$

Then

$$A = \frac{N}{Z_\beta}.$$

Hence, the most probable distribution, i.e., the Maxwell-Boltzmann distribution, is given by

$$\frac{N_i}{N} = \frac{1}{Z_\beta} g_i e^{-\beta E_i}, \quad i = 1, \dots, n.$$

By Boltzmann's entropy formula (12) in Statistical Mechanics, the thermodynamic entropy of the macrostate system is given by

$$S = k \log W.$$

Using the explicit expression of the Maxwell-Boltzmann distribution, one can prove that

$$S = kN \log Z_\beta + k\beta E.$$

Let

$$\langle E \rangle = \sum_{i=1}^n \frac{n_i E_i}{N}.$$

Then we obtain the Boltzmann statistical mechanics interpretation of the S -entropy:

$$S = \lim_{N \rightarrow \infty} \frac{\log W}{N} = \log Z_\beta + \beta \langle E \rangle.$$

In the continuous case, let \mathbb{P} be the Gibbs measure on a canonical ensemble (Ω, \mathcal{F}) with Hamiltonian energy function $E : \Omega \rightarrow \mathbb{R}^+$. Then the Maxwell-Boltzmann distribution is given by the following probability measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$

$$dP(E) = \frac{e^{-\beta E}}{Z_\beta} g(E) dE,$$

where $\beta \in \mathbb{R}$ is the Boltzmann constant, $g(E)dE$ denotes the “density of states” measure, whose physical meaning is the number of microstates with energy levels in the range $[E, E + dE]$, and

$$Z_\beta = \int_{\mathbb{R}^+} e^{-\beta E} g(E) dE$$

is the partition function. In this case, one can define the temperature of the macrostate system by

$$T = \frac{1}{\beta},$$

and define the Helmholtz free energy by

$$F = -\frac{1}{\beta} \log Z_\beta.$$

One can verify that, the average of the energy with respect to the Maxwell-Boltzmann distribution satisfies

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z_\beta,$$

and the following formula holds for the Boltzmann entropy S of the Maxwell-Boltzmann distribution:

$$S = \log Z_\beta - \beta \frac{\partial}{\partial \beta} \log Z_\beta.$$

Moreover, the fluctuation of the energy with respect to the Maxwell-Boltzmann distribution is given by

$$\sigma := \langle (E - \langle E \rangle)^2 \rangle = \frac{\partial^2}{\partial \beta^2} \log Z_\beta,$$

and the derivative of the entropy with respect to β satisfies

$$\frac{\partial S}{\partial \beta} = -\beta \frac{\partial^2}{\partial \beta^2} \log Z_\beta = -\beta \sigma.$$

By the definition formula $\beta = \frac{1}{T}$, we can verify the following formulas

$$\begin{aligned} \langle E \rangle &= T^2 \frac{\partial}{\partial T} \log Z_\beta, \quad F = -T \log Z_\beta, \\ S &= \frac{\partial}{\partial T} (T \log Z_\beta) = -\frac{\partial F}{\partial T}, \end{aligned}$$

and

$$\frac{\partial S}{\partial T} = \frac{\sigma}{T^3}.$$

3 Perelman’s interpretation of the W -entropy for Ricci flow

In Section 5 in [31], Perelman gave a heuristic interpretation for the W -entropy using statistical mechanics. Let $(M, g(\tau))$ be a family of closed Riemannian manifolds, $dm(\tau) = (4\pi\tau)^{-n/2} e^{-f(\tau)} dv_{g(\tau)}$ a probability measure on $(M, g(\tau))$, where $g(\tau)$ satisfies the Perelman’s backward Ricci flow equation $\frac{\partial}{\partial \tau} g_{ij} = R_{ij} + \nabla_i \nabla_j f$, and $f(\tau)$ satisfies the heat

equation $\partial_\tau f = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}$. Assume that for some canonical ensemble $(\Omega, \mathcal{F}, E, \mathbb{P})$ on which there is a “density of state measure” $g(E)dE$ such that the partition function $Z = \int_{\mathbb{R}^+} e^{-\beta E} g(E)dE$ is given by

$$\log Z = \int_M (-f + \frac{n}{2}) dm, \quad (14)$$

where $\beta = \frac{1}{\tau}$, and τ is regarded as the temperature. Then, using the formulas in statistical mechanics, we can obtain

$$\begin{aligned} \langle E \rangle &= -\tau^2 \int_M (R + |\nabla f|^2 - \frac{n}{2\tau}) dm, \\ S &= - \int_M (\tau(R + |\nabla f|^2) + f - n) dm, \\ \sigma &= 2\tau^4 \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 dm. \end{aligned}$$

Alternatively, we can consider the evolution equations (1) and (2) by replacing the t -derivatives by minus of the τ -derivatives, where $\tau = T - t$. This implies that

$$W = -S, \quad (15)$$

and

$$\frac{dW}{dt} = 2 \int_M \tau \left| Ric + \nabla^2 f - \frac{g}{2\tau} \right|^2 dm. \quad (16)$$

The above statistical mechanics interpretation (15) of the W -entropy and the derivation of the derivative formula (16) are heuristic. The problem whether there is a canonical ensemble $(\Omega, \mathcal{F}, E, \mathbb{P})$, equivalently, whether there is a “density of states” measure $g(E)dE$, such that the partition function $Z = \int_{\mathbb{R}^+} e^{-\beta E} g(E)dE$ satisfies Perelman’s required condition $\log Z = \int_M (\frac{n}{2} - f) dm$ remains open. It is a very interesting to know whether one can use some ideas of the quantum field theory to construct a “density of states” measure on some natural canonical ensemble such that the Boltzmann entropy of the corresponding Maxwell-Boltzmann distribution is exactly given by $S = -W$. See Section 5 below.

4 A probabilistic interpretation of the W -entropy for Ricci flow

In [21], the author gave a probabilistic interpretation of Perelman’s W -entropy for the Ricci flow. We now present this interpretation. For this, observe that

$$\int_M (\frac{n}{2} - f) dm = \frac{n}{2} (1 + \log(4\pi\tau)) - H(m),$$

where

$$H(m) = - \int_M u \log u dv = \int_M (f + \frac{n}{2} \log(4\pi\tau)) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv$$

is the Boltzmann-Nash-Shannon entropy of the probability measure $dm = u dv$, where $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$. On the other hand, let

$$d\gamma_n(x) = \frac{e^{-\frac{\|x\|^2}{4\tau}}}{(4\pi\tau)^{n/2}} dx$$

be the Gaussian measure on \mathbb{R}^n . Then it is well-known that the Boltzmann-Nash-Shannon entropy of γ_n is given by

$$H(\gamma_n) = \frac{n}{2}(1 + \log(4\pi\tau)).$$

Hence, $\log Z = H(\gamma_n) - H(m)$ is the difference of the Boltzmann-Nash-Shannon entropy of the Gaussian measure γ_n on \mathbb{R}^n and the Boltzmann-Nash-Shannon entropy of the heat kernel measure $dm = u(\tau)dv_{g(\tau)}$ on $(M, g(\tau))$. In view of this, we have the following probabilistic interpretation of the W -entropy for the Ricci flow

$$W = \frac{d}{d\tau}(\tau(H(\gamma_n) - H(u)),$$

and

$$\frac{dW}{d\tau} = 2\tau \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 dm.$$

Remark 4.1 During the preparation of this paper, Songzi Li pointed out to the author that the quantity $F := -\tau \log Z$ appeared in the definition formula of the W -entropy $W = -\frac{\partial F}{\partial \tau}$ is actually the Helmholtz free energy function on the canonical ensemble $(\Omega, \mathcal{F}, E, \mathbb{P})$, where $\tau = \frac{1}{\beta}$ is regarded as the temperature. According to Otto [27], if one chooses $F(u) = \int_M u \log u dv + \int_M V u dv$ as the Helmholtz free energy of a certain thermodynamic system, where $V \in C^2(M)$, then the gradient flow of F in the Wasserstein space equipped with a suitable infinite dimensional Riemannian metric is the Fokker-Planck equation $\partial_t u = \Delta u + \nabla \cdot (u \nabla V)$. In view of this, the W -entropy for the Ricci flow can be regarded as the thermodynamic entropy corresponding to the Helmholtz free energy function $F(u) = \tau(H(\gamma_n) - H(u))$, which is the difference of the free energy of a certain thermodynamic system on $(M, g(\tau))$ and the free energy of the Maxwell thermodynamic equilibrium state on \mathbb{R}^n . This explanation is indeed equivalent to the above mentioned Perelman's heuristic interpretation, and is very close to the kinetic interpretation given by the author in [22] for the W -entropy for the Fokker-Planck equation on complete Riemannian manifolds.

5 Comparison between Boltzmann's H -theorem and Perelman's entropy formula

In the above sections, we have discussed, following the idea of Perelman, the relationship between the Boltzmann H -entropy and the Perelman W -entropy. In this section we would like to compare the Boltzmann H -theorem and the Perelman W -entropy formula.

The Boltzmann H -entropy formula (8) and the Perelman W -entropy formula (5) are in the same spirit in the following three points.

(i) The Boltzmann H -entropy formula (8) gives the time derivative formula of the Boltzmann H -entropy along the solutions of the Boltzmann equation, and the Perelman W -entropy formula (5) gives the time derivative formula of the Perelman W -entropy along the solutions of the Ricci flow equation and the conjugate heat equation.

(ii) From the Boltzmann H -entropy formula (8) and the Perelman W -entropy formula (5), we can easily derive the monotonicity of the Boltzmann H -entropy along the Boltzmann equation and the monotonicity of the Perelman W -entropy along the Ricci flow and the conjugate heat equation.

(iii) We can derive the equation of the equilibrium state for the Boltzmann H -entropy and for the Perelman W -entropy. The equilibrium state of the Boltzmann H -entropy satisfies the equation (10), and the equilibrium state of the Perelman W -entropy is the shrinking Ricci solitons which satisfy the equation (6).

In the case of the Boltzmann equation, the problem of the convergence rate of the solutions of the Boltzmann equation towards the equilibrium state (i.e., the Maxwell distribution) is the well-known Cercignani conjecture in the study of the Boltzmann equation. See [11, 6, 7, 15, 33]. From the above comparison, we can raise the following interesting problem for the further study of the Perelman's W -entropy.

Problem 5.1 *What is the longtime behavior of the evolution equations (1) and (2)? If the solution $(g(t), f(t))$ of (1) and (2) exists on $[0, \infty)$, what is the convergence behavior of $(g(t), f(t))$ towards its limit as $t \rightarrow \infty$?*

In [31], Perelman raised the following question

Problem 5.2 *If the flow is defined for all sufficiently large τ (that is, we have an ancient solution to the Ricci flow, in Hamilton's terminology), we may be interested in the behavior of the entropy S as $\tau \rightarrow \infty$. A natural question is whether we have a gradient shrinking soliton whenever S stays bounded.*

6 An open problem: the “density of states” measure for the Ricci flow

Following Perelman's heuristic interpretation of the W -entropy for the Ricci flow, we would like to discuss the following open problem.

Problem 6.1 Can one prove the existence of a certain canonical ensemble (Ω, \mathcal{F}) with *a priori* measure $D\omega$ and a Hamiltonian function $E : \Omega \rightarrow \mathbb{R}^+$ such that the partition function of the corresponding Gibbs measure \mathbb{P} on (Ω, \mathcal{F}) satisfies Perelman's condition

$$\log \int_{\Omega} e^{-\beta E(\omega)} D\omega = \int_M \left(\frac{n}{2} - f \right) dm?$$

Equivalently,

$$\log \int_{\Omega} e^{-\beta E(\omega)} D\omega = H_n(u, t),$$

where

$$H_n(u, t) := \frac{n}{2}(1 + \log(4\pi\tau)) - \int_M u \log u dv.$$

Moreover, if such a canonical ensemble $(\Omega, \mathcal{F}, \mathbb{P})$ and a Hamiltonian function $E : \Omega \rightarrow \overline{\mathbb{R}^+}$ exist, how to construct them in a natural way?

Suppose that such a canonical ensemble (Ω, \mathcal{F}) with a Hamiltonian function $E : \Omega \rightarrow \mathbb{R}^+$ exist. Let \mathbb{P} be the Gibbs measure on (Ω, \mathcal{F}) . Let $dP(E)$ be the image measure of the Gibbs measure \mathbb{P} on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ under $E : \Omega \rightarrow \mathbb{R}^+$, in other words, $dP(E)$ is the probability law of the Hamiltonian energy function E considered as a random variable from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Then one can write

$$dP(E) = \frac{e^{-\beta E}}{Z_{\beta}} g(E) dE,$$

where dE is the Lebesgue measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, and $g(E)dE$ is the “density of states” measure of the canonical ensemble $(\Omega, \mathcal{F}, \mathbb{P})$ and the Hamiltonian $E : \Omega \rightarrow \mathbb{R}^+$. In view of this, the “density of states” measure $g(E)dE$ must satisfy

$$\log \int_{\mathbb{R}} e^{-\beta E} g(E) dE = H_n(u, t). \quad (17)$$

This is to say, the log-Laplace transformation of the “density of states” measure $g(E)dE$ on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ equals to $H_n(u, t) = \int_M \left(\frac{n}{2} - f\right) dm$. Formally, by analytic extension of the partition function $\beta \rightarrow Z(\beta) := \int_{\mathbb{R}^+} e^{-\beta E} g(E) dE$, and using the inverse Fourier transformation, we have (cf. [30] p. 56)

$$\begin{aligned} g(E) &= \frac{1}{2\pi i} \int_{\beta' - i\infty}^{\beta' + i\infty} e^{\beta E} Z(\beta) d\beta \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(\beta' + i\beta'')E} Z(\beta' + i\beta'') d\beta'', \end{aligned}$$

where $\beta = \beta' + i\beta''$ is a complex variable with $\beta' > 0$, and the path of integration is parallel to the right of the imaginary axis, i.e., along the straight line $\operatorname{Re} \beta = \beta' > 0$. Of course, the path may be continuously deformed so long as the integral converges. Thanks to this observation, it might be possible for us to prove the existence of the “density of states” measure $g(E)dE$ on a natural canonical ensemble $(\Omega, \mathcal{F}, \mathbb{P})$ and for a natural Hamiltonian $E : \Omega \rightarrow \mathbb{R}^+$ such that the partition function of the Maxwell-Boltzmann distribution on $(\Omega, \mathcal{F}, E, \mathbb{P})$ satisfies Perelman’s condition (17), i.e.,

$$\log \int_{\mathbb{R}} e^{-\beta E} g(E) dE = H_n(u, t).$$

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